New Upper Bounds on Rubik's cube

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Abstract

In 1995 Michael Reid proved that Rubik's cube can be solved in at most 29 moves when considering the face turn metric and at most 42 moves when considering the quarter turn metric; see [Reid, 1995a]. In this paper we will generalize these methods to get an upper bound of 27 in the face turn metric and 34 in the quarter turn metric.

Keywords: Rubik's cube, upper bound, diameter

1 Introduction

Michael Reid proved that Rubik's cube can be solved in at most 29 moves when considering the face turn metric and at most 42 moves when considering the quarter turn metric see; [Reid, 1995a]. In this paper we will generalize these methods to obtain an upper bound of 27 in the face turn metric and 34 in the quarter turn metric. Since all upper bounds known so far are based on computations, many arguments are verified by direct computation as will be seen. If the reader is interested in more details on how these computations were done we refer to the references and the cube forums [Fo,] and [Fo2,].

Recently in [Kunkle and Cooperman, 2007] one of these bounds has been improved to 26 [1]. For the proof the authors used 8000 CPU hours of computation time, while our method for the 27 proof is much more economic (10 CPU hours). In addition we can supply some additional data that allows to verify the proof automatically in matter of a few CPU seconds.

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[1] A gap was found in this paper; see [Fo,]

In Section 2 we present some basic theory and terminology that will be used throughout this paper. In Section 3 we present the difference between our method and Reid's, moreover we give an improvement of the upper bound in the quarter turn metric. Sections 4, 5, 6 are about the bounds in the quarter turn metric but describe some methods that also work in the face turn metric. Sections 7, 8, 9 are about the proof in the face turn metric. In section 10 we comment on possible future work.

2 Some Basic Theory

It is well known that Rubik's cube can be modeled in terms of a group. A brief description how this group and its elements can be specified is as follows:

Fix a random face of the cube and label it with the letter U. Take the face opposite to U and label it with D. Take one random face of the 4 unlabeled faces left and label it with L. Label the opposite face to L with R. Finally take one of the remaining two unlabeled faces and label it with F and the last face with B. One can verify that given two cubes, labeled in the way specified above, are the same up to rotations and mirror reflections. The letters we used to label the faces are called by cubists Up, Down, Left, Right, Front, Back.

Next we use these labels to define Rubik cube moves. When we say that we have done move U to a cube we mean that we take the physical cube with face U towards us and rotate it 90 degrees clockwise. Move U^{-1} means 90 degrees counter-clockwise rotation, i.e., undo U. As usual we write U^2 for the move sequence UU. The same conventions apply to the other moves. The Rubik cube group G then is defined by the group generated by these moves. A group element will then be represented by a sequence of moves e.g. $UD^{-1}L^2$. This representation is not unique since $UD = DU$. Throughout this paper we write X for the generating set $\langle U, L, F, B, R, D\rangle$ and 1_G for the identity element.

Let G' be a group and H' a subgroup of G'. Let X' be the generating set of G'. We will employ the notion of a Cayley graph associated to (G', X') and Schreier coset graph associated to (G', H', X') for our purpose. The Cayley graph is defined as the graph whose nodes are the elements of G', and given two nodes g_1, g_2 there is an edge between them iff there exists an $x \in X' \cup X'^{-1}$ such that $g_1 x = g_2$. The Schreier coset graph is defined as the graph whose nodes are the right cosets $H' \backslash G'$ and given two nodes $H'g_1, H'g_2$ there is an edge between them iff there exists an $x \in X' \cup X'^{-1}$ such that $H'g_1 x = H'g_2$.

We identify the Cayley graph with (G', X') and the Schreier coset graph with (G', H', X').

Note that the usual definition of these graphs does not require that the generating set is closed under inversion. In this paper this will always be the case.

We also need the notions of length and distance in a graph. A path between two nodes g_1, g_2 in a graph is a sequence of nodes $g_1 = x_1, x_2, \ldots, x_n = g_2$ such that there is an edge between x_k and x_{k-1} for $k = 1, \ldots, n$. The length of this path is defined to be n. The distance between g_1 and g_2 is defined as the minimum length among all paths between g_1 and g_2.

If we assign to each edge in a graph a positive number called weight, then we get a weighted graph. If (a, b) is an edge between the nodes a and b, we write $w(a, b)$ for the weight of (a, b). A path between two nodes g_1, g_2 in a weighted graph is a sequence of nodes $g_1 = x_1, \ldots, x_n = g_2$ as in the unweighted situation, and the length of such a path is defined to be $\sum_{k=1}^{n-1} w(x_k, x_{k+1})$. Again the distance between g_1 and g_2 in a weighted graph is defined to be the minimum length among all paths between g_1 and g_2.

By abuse of language, by the distance of a node g in a Cayley graph or a Schreier coset graph, we mean the distance between g and the identity element of the group, respectively the identity coset.

The diameter of a Cayley graph and Schreier coset graph is defined to be the maximum distance among all nodes of the graph.

We reserve the letter G for the Rubik's cube group and the letter H for the subgroup generated by $\langle U, D, L^2, F^2, B^2, R^2 \rangle$.

We show how to compute upper bounds for the diameter of the Cayley graph (G, X), also called the quarter turn metric, henceforth abbreviated with QTM. And we will also show how to compute upper bounds for the diameter of the Cayley graph $(G, X \cup X^2)$ also known as the face turn metric, henceforth abbreviated with FTM.

When we are in QTM then our moves will be the elements of $X_q := X \cup X^{-1}$. In FTM our moves will be the elements of $X_f := X \cup X^{-1} \cup X^2$.

Assume we are in QTM. When we say that a given $g \in G$ can be solved in n moves, we mean that $g = x_1 \cdot \ldots \cdot x_m$ for some $x_1, \ldots, x_m \in X_q$ and $m, n \in \mathbb{N}$ with $m \leq n$.

Analogously assume we are in FTM. When we say that a given $g \in G$ can be solved in n moves, we mean that $g = x_1 \cdot \ldots \cdot x_m$ for some $x_1, \ldots, x_m \in X_f$ and $m, n \in \mathbb{N}$ with $m \leq n$.

Given a generating set X' we write $d_{X'}(g)$ for the distance of g in the Cayley graph (G, X'). Analogously we write $d_{X'}(Hg)$ for the distance of Hg in the Schreier coset graph (G, H, X'). We write $d(Hg)$, respectively $d(g)$, if X' is clear from the context.

A distance record for a subset of nodes S in a given Cayley graph (G', X'), resp. Schreier coset graph (G', H', X'), is a list $(d_0, d_1, \ldots, d_k)$ where d_j is the number of nodes g, resp. Hg, in S at distance j; i.e. $d_j = |\{g \in S | d_{X'}(g) = j\}|$ resp. $d_j = |\{Hg \in S | d_{X'}(Hg) = j\}|$. By abusing the language, we write distance record for (G', X') resp (G', H', X') if the set S is G' resp if the set S is $H' \backslash G'$. In the tables we abbreviate distance record by "d.r.". write d.r. for distance record.

Let X' be a generating set. The basic method for computing upper bounds is first to compute a distance record for (G, H, X') and then to compute a distance record of H in (G, X'). Taking the sum of the maximum distance in each record gives a first upper bound. But as described below, there are several methods to improve the bounds.

3 Computing Distances

There are some differences between our approach and Reid's approach. Reid computes the distances in the Cayley graph of the group H with respect to generators $X_H = < U, D, L^2, F^2, B^2, R^2 >$.

In the quarter turn metric he considers a weighted Cayley graph (H, X_H) where an edge (g_1, g_2) has weight 2 if $g_1 x = g_2$ for $x \in \{L^2, F^2, B^2, R^2\}$ and weight 1 otherwise. Note that the weight of (g_2, g_1) is the same because of $g_2 x^{-1} = g_1$.

Note that one node $h \in H$ can appear in the graph (H, X_H) as having a distance l, while the distance of h in the Cayley Graph (G, X) might be strictly less than l, due to a possibly increased number of edges. Note that (H, X_H) is not a subgraph of (G, X); for instance , $(1_G, L^2)$ is an edge (of weight 2) in (H, X_H) but not in (G, X). In (G, X) we only find the path $(1_G, L, L^2)$.

One idea to lower the upper bound is to compute the distance of the elements of the group H in (G, X). One can see the difference by studying Table 1.

Table 1 has to be interpreted in the following way:

Given a perturbated Rubik cube, one first determines a suitable representation of the corresponding element $g' \in G$. For example as in [Schoenert,]. Next, determine the coset Hg such that $g' \in Hg$. Herbert Kociemba shows on his home page [Ko,] how one can represent cosets of H. Next, we look up the distance of the coset Hg. To do so, the cosets together with their distances (as in column 3 in Table 1) have to be stored in a computer. In this way we know the distance $d_X(Hg)$.

Next we apply moves to it, i.e., Hgx for $x \in X_q$ such that $d(Hgx) < d(Hg)$ until we arrive at the identity coset $Hgx_1 x_2 \ldots x_m = H$. Consequently, for $g' = hg$ we have:

$$g' x_1 x_2 \ldots x_m \in H$$

Table 1:

Distance	d.r. of H in (H, X_H) [Reid, 1995c]	d.r. of H in (G, X) [Radu, 2006a]	d.r. of (G, H, X) [Reid, 1995b]
0	1	1	1
1	4	4	4
2	10	10	34
3	36	36	312
4	123	123	2,772
5	368	368	24,996
6	1,192	1,320	225,949
7	3,792	4,800	2,017,078
8	11,263	15,495	17,754,890
9	34,352	54,016	139,132,730
10	102,638	194,334	758,147,361
11	287,320	656,752	1,182,378,518
12	810,144	2,222,295	117,594,403
13	2,261,028	7,814,000	14,072
14	5,941,838	26,402,962	
15	16,291,708	89,183,776	
16	41,973,415	297,590,924	
17	107,458,884	929,624,528	
18	269,542,476	2,573,889,614	
19	628,442,876	5,506,671,444	
20	1,367,654,200	6,551,983,325	
21	2,613,422,312	3,219,955,376	
22	3,997,726,648	301,913,989	
23	4,444,701,268	249,300	
24	3,661,653,732	8	
25	1,906,936,668		
26	407,132,392		
27	34,358,944		
28	1,664,168		
29	14,840		
30	160		
	19,508,428,800	19,508,428,800	2,217,093,120

and column 2 in Table 1 shows:

$$d_X(g'x_1x_2\dots x_m) \le 24$$

Because of $m \le 13$ (owing to the distance record of $H\backslash G$) we get an upper bound of 24+13.

One can also formulate all this as follows:

Given a $g' \in G$ we can always write it as $g' = ht$ for some $h \in H, t \in G$ with $d_X(h) \le 24$ and $d_X(t) \le 13$.

4 The Symmetry Group

Let M be the group of space symmetries of the cube which contains rotations and reflections. This group has 48 elements, and its natural action on the cube induces a group action on the Rubik cube in an obvious way. This action then gives rise to an action on G which is based on the following fact: Namely, if X_q are the generators of Rubik's cube group and their inverses, then $m \in M, x \in X_q$ implies $m^{-1}xm \in X_q$. Especially if $g \in G$ and $m \in M$ then $d_{X_q}(m^{-1}gm) = d_{X_q}(g)$. A similar relation holds for X_f. That is if $g \in G$ and $m \in M$ then $d_{X_f}(m^{-1}gm) = d_{X_f}(g)$.

Hence this conjugation action of M defines in this way an equivalence relation on the cube group G. I.e., two elements are equivalent if they are M conjugate. Consequently, M can be viewed as the symmetry group of G that leaves distances invariant. More about the symmetry group can be found at [Hoey and Saxe, 1980].

5 Rubik's Cube can be Solved in 35 Moves

We will first treat the case of QTM, so unless otherwise stated, we assume that our generating set is X_q.

The group H defined above is not invariant under the whole symmetry group M, but is invariant under a subgroup M' of M that has order 16. This can easily be verified by computation with GAP [Ga,].

Observation 1

All the elements of H that have distance 24 in Table 1, column 2, all lie in one equivalence class under M'. This again is verified with GAP. A representative of this equivalence class is:

$$h_{24} := UB^2UB^2D^{-1}F^2DL^2UF^2U^{-1}F^2UR^2F^2U^{-1}.$$

As a side remark, it should be mentioned that this generator expression was given by Bruce Norskog in a forum message [Fo,], it has the beautiful property that it is minimal in both FTM and QTM.

Observation 2

Typically elements with distance close to the diameter are local maxima. Local maxima are defined as follows:

In QTM $g \in G$ is said to be a local maximum if $d(gx) \leq d(g)$ for all $x \in X_q$. I.e., all neighbouring nodes of g have lower distance.

Local maxima have an interesting property that we will use, namely if $g \in G$ is a local maxima then for all $g' \in G$ we have $d(gg') \leq d(g) + d(g') - 2$.

Note that the property of being a local maxima is preserved by the group M i.e., if g is a local maxima then so is $m^{-1}gm$ for all $m \in M$.

We explained at the end of Section 3 that any $g \in G$ may be written as ht for $h \in H$ and $t \in G$ with $d(h) \leq 24$ and $d(t) \leq 13$.

Moreover, we observe that all $h \in H$ have distance strictly less than 24 except the orbit of h_{24} i.e. $h_{24}^{M'}$.

Finally, given an arbitrary $g \in G$ with $g = ht$ we have two possibilities for h, namely:

1. $h \in H \backslash h_{24}^{M'}$, or
2. $h \in h_{24}^{M'}$.

In the first case we have that $d(ht) \leq 23 + 13$.

In the second case we can verify that q_{24} is a local maxima which implies that every element in $h_{24}^{M'}$ is a local maxima. Consequently $d(ht) \leq 24 + 13 - 2$ in this case.

By these simple observations we have proven that Rubik's cube is solvable in 36 moves.

Observation 3:

Another interesting observation is that the cube group G has a subgroup N' of index 2. This can be verified by asking GAP to perform a normal subgroup computation of G. We have the canonical homomorphism $\phi : G \mapsto G/N'$.

Because G/N' has order 2 we have an isomorphism $\psi : G/N' \mapsto \{1, -1\}$.

For $g \in G$ we say that g is odd, if $\psi(gN') = -1$, otherwise it is called even.

It is important to note that all elements in X_q are odd. Let $n \in \mathbb{N}$, then $d_X(g) = 2n + 1$ implies g is odd, and $d_X(g) = 2n$ implies that g is even.

We verify with GAP that $N := H \cap N'$ is a subgroup of index 2 in H. We have computed the distance record of (G, N, X); see Table 2.

Table 2:

Distance	d.r. of (G, N, X) [Radu, 2006b]
0	1
1	9
2	68
3	624
4	5544
5	49992
6	451898
7	4034156
8	35109780
9	278265460
10	1516294722
11	2364757036
12	235188806
13	28144

If the distance record of H in (G, X) is $(d_1, d_2, d_3, \ldots, d_{24})$ then the distance record of N in (G, X) is $(d_1, 0, d_3, 0, \ldots, d_{24})$.

By the same argumentation as at the end of Section 3, we conclude that any $g \in G$ may be written as $g = nt$ for $n \in N$ and $t \in G$ with $d(n) \leq 24$ and $d(t) \leq 13$. But by observation 2 the $n \in N$ with $d(n) = 24$ are local maxima and for such n we have that $d(nt) \leq 24 + 13 - 2$. And since no element of N has distance 23 (otherwise n would be odd) we conclude that Rubik's cube can be solved in 35 moves.

6 Refinement:Rubik's Cube can be Solved in 34 Moves

In Section 3 we described how to get an upper bound for the distance of any perturbed cube corresponding to a $g \in G$. Whenever Ng is a distance 13 coset we will find another method of providing a bound for g. If this new method always can give a bound of 34 or less on the distance of g then we have proven that Rubik's cube is solvable in 34 moves. This can be achieved simply by taking each such g and trying to express it as a word in generators of length 34 or less. Exactly this technique has been used by Bruce Norskog [Norskog, 2005] to prove that Rubik is solvable in at most 38 moves. In his proof he worked with a group smaller than N and with fewer cosets. In our situation this method will not work since we have two many cases to consider and the computation will not take reasonable time.

The method we propose is the following:

Recall that each element in $g^M := \{m^{-1}gm | m \in M\}$ has the same distance as g.

It is thus sufficient to show that for at least one element $g' \in g^M$ the coset Ng' has distance less then 13.

Instead of running this procedure for each element g with $d(Ng) = 13$ we are executing it for a whole coset of Tg where

$$T := \bigcap_{m \in M} m^{-1}Nm.$$

We then have an action of M on the cosets of T given by $m \cdot Tx := Tm^{-1}xm$. Note that this is well defined since T is invariant under M.

The set $T_{13} := \{Tx | d(Nx) = 13\}$ gives us all cosets of T contained in a distance 13 coset of N. We take each $Tx \in T_{13}$ and do a computer check to see if there exists an element in its orbit under the action $\cdot$ that is contained in a coset of N at distance strictly less then 13. If this is the case then each element in this coset is solvable in 34 moves or less.

From the respective computation we can conclude that there are only 120 cosets of T for which no element of their orbits is contained in a coset of N of distance strictly less than 13.

As a consequence, we need to show that also the elements of these 120 cosets are solvable in 34 moves. To this end we only need to show it for a coset in the orbit under $\cdot$ of each such coset. The group M also acts on the subset of $T\backslash G$ consisting of these 120 cosets, and this restricted action gives rise to 12 orbits. We selected a complete set of representatives from each of these orbits and computed distance records for them. From the distance records we concluded that all the elements in these cosets are solvable in 23 moves maximum. The distance records are found in Table 3 and should be interpreted as follows:

Let

$$T_H := \bigcap_{m \in M} m^{-1}Hm.$$

The group T_H contains the group T as a subgroup of index 2. GAP computations show that each of these 12 cosets are contained in 6 cosets $R_1, R_2, R_3, R_4, R_5, R_6$ of the group T_H from where we choose $r_1, r_2, r_3, r_4, r_5, r_6$ as representatives.

The explicit form of this representatives is as follows:

$r_1 := DF^2L^2F^2UR^2DR^{-1}D^{-1}U^2F^2L^{-1}FD^2B^{-1}L^2DBU$

$r_2 := B^2LU^2F^2LB^2L^{-1}F^{-1}D^2F^{-1}DFUR^{-1}BU^2L^2D^{-1}L^{-1}F^2$

$r_3 := R^2B^2FD^2B^{-1}U^2L^2U^{-1}B^2L^{-1}B^{-1}D^2BRD^2BU^2LF^2D^{-1}$

$r_4 := L^2FU^2F^{-1}D^2L^2R^{-1}UB^2D^{-1}U^2R^2F^{-1}D^{-1}R^{-1}U^2LF^{-1}L^{-1}F^{-1}$

$r_5 := B^2F^2L^{-1}F^2L^2B^2DR^2D^{-1}L^{-1}R^{-1}FU^{-1}RB^{-1}L^2D^{-1}B^{-1}D^2R^{-1}$

$r_6 := B^2L^{-1}F^2LB^2L^2DF^{-1}LF^2UB^2R^{-1}BD^{-1}RD^2BRU^2$

In conclusion, we have shown that Rubik's cube can be solved in at most 34 moves.

Table 3:

dist	d.r. of R_1	d.r. of R_2	d.r. of R_3	d.r. of R_4	d.r. of R_5	d.r. of R_6
15	0	0	0	0	0	8
16	140	96	84	32	96	72
17	552	560	882	340	2424	1162
18	15392	13670	6099	8398	9334	8950
19	100132	107810	110664	82378	158595	135454
20	979199	1870166	747573	826886	878493	868592
21	1872684	977736	1866150	1883574	1820223	1841298
22	995925	999154	1236900	1155340	1102733	1113072
23	17288	12120	12960	24364	9414	12734

Note:

The fact that each group element in the above cosets has distance less than 23 is easily verified by providing for each element of the cosets (3.981.312 for each coset) a representation where the elements are written as product of generators. This verification can be made in a couple of seconds using a normal modern computer. The cosets resulting from the procedure described above can also be identified in just a matter of minutes using a computer program such as GAP.

7 The Face Turn Metric

In the face turn metric FTM, our generating set will be X_f, unless otherwise stated.

It turns out that by using similar methods one can achieve a lowering of the upper bound of 29 for the face turn metric. However, not all the methods we used in QTM work equally well for FTM.

Michael Reid's upper bound is based on the group H here also. Let

$$X_{H_f} := \{U, D, U^2, L^2, F^2, B^2, R^2, D^2\}$$

He computes a distance record of (G, H, X_f) and a distance record of (H, X_{H_f}). Note that (H, X_{H_f}) is not a weighted graph here. The distance records can be seen in Table 4.

Table 4:

Distance	d.r of (H, X_{H_f}) [Reid, 1995c]	d.r. of H in (G, H, X_f) [Rokicki, 2006]	d.r. of (G, H, X_f) [Winter, 1992]
0	1	1	1
1	10	10	4
2	67	67	50
3	456	456	592
4	3,079	3,079	7,156
5	19,948	20,076	87,236
6	123,074	125,218	1,043,817
7	736,850	756,092	12,070,278
8	4,185,118	4,331,124	124,946,368
9	22,630,733	23,639,531	821,605,960
10	116,767,872	122,749,840	1,199,128,738
11	552,538,680	582,017,108	58,202,444
12	2,176,344,160	2,278,215,506	476
13	5,627,785,188	5,790,841,966	
14	7,172,925,794	7,240,785,011	
15	3,608,731,814	3,319,565,322	
16	224,058,996	145,107,245	
17	1,575,608	271,112	
18	1,352	36	

By the same arguments we used before we get here an upper bound of 30. As can be seen by comparing the distance record of H in (G, X_f) with the distance record of (H, X_{H_f}), it is not possible to get any improvement on the upper bound, since the maximal distance is 18 in both cases. For QTM on the other hand, the maximal distance differed by 6. Consequently we obtain an upper bound of 30.

By $d_H(x)$ we mean the distance of x in the (H, X_{H_f})

Let

$$Y := \{F, B, L, R, F^{-1}, B^{-1}, L^{-1}, R^{-1}\}.$$

Table 4 together with the discussion at the end of Section 3 shows that each $g \in G$ may be written as $g = h \cdot t$ with $h \in H$ and $t \in G$ such that $d(h) \leq 18$ and $d(t) \leq 12$. When $d(t) > 1$ we can always write $t = yx$ with $y \in Y$ and $d(x) = d(t) - 1$ where $d(h) \leq 18$ and $d(x) \leq 11$ or $t = 1_G$ and $d(h) \leq 18$. Clearly this statement is true if we assume $y \in X_f$. But if $y \notin Y$ then $hy \in H$ and $d(hy) \leq 18$. Applying this procedure several times we end up with either $y \in Y$ or $t = 1_G$.

Reid showed that for each h (1352 in total) with $d_H(h) = 18$ and for each $k \in \{F^2, B^2, L^2, R^2\}$ we have that $d_H(hk) = 17$.

In the next step we take an element $g = ht$ with $d_H(h) = 18$ and $d(t) = 12$ then $ht = (hy^2)(y^2)yx = (hy^2)y^{-1}x$ with $y \in Y$. Note that $y^4 = 1_G$. Since $d_H(hy^2) = 17$ by assumption and $d(x) = 11$ we get that $d(ht) \leq 29$. Consequently Rubik's cube can be solved in 29 moves.

This method can be extended to get an upper bound of 28 as it will be seen in next Section.

8 Rubik's Cube can be Solved in 28 Moves

Given $g \in G$ rewrite it as $ht = g$ with $h \in H, t \in G$ such that $d(t) \leq 12$, $d_H(h) \leq 18$. Clearly, Rubik's cube can be solved in 28 moves if $d_H(h) \leq 16$. So we only need to consider the cases when $d_H(h) = 17$ or $d_H(h) = 18$.

- Case $d_H(h) = 18$

  We may assume that $d(t) \geq 4$, otherwise there is nothing to prove. Then we can write $t = x_1x_2x_3x', d(x') = d(t) - 3, x_2, x_3 \in X_f, x_1 \in Y$. By substituting for t we get $g = hx_1x_2x_3x'$. We have checked by computer using Herbert Kociemba's Cube Explorer [Ko,], that if $h \in H$ with $d_H(h) = 18$ then for all $x_1 \in Y, x_2 \in X_f$ and $x_3 \in X_f$ one of the following estimates hold:

  1. $d(hx_1) \leq 17$;

2. $d(hx_1x_2) \leq 18$;
3. $d(hx_1x_2x_3) \leq 19$.

Just by looking at the formulas one can easily verify that $d(g) \leq 18 + d(t) - 2$.

- Case $d_H(h) = 17$.

1. Subcase $d(hy'^2) \leq 16$ for all $y' \in Y$

   We can assume that $d(t) \geq 2$, otherwise we are done. Write $t = yx$ with $y \in Y$ and $d(x) = d(t) - 1$.

   Then we can write $g = hy^2y^2yx = hy^2y^{-1}x$. Because $d(hy^2) \leq 16$ we get that $d(g) \leq 16 + 1 + d(t) - 1$.

   Remark 1:

   Many elements h fall in this subcase and the computer can check this condition very fast.

2. Subcase there exists an $y \in Y$ such that $d_H(hy^2) > 16$.

   W.l.o.g we assume $d(t) \geq 4$. Write $t = x_1x_2x_3x', x_1 \in Y, x_2, x_3 \in X_f, d(x') = d(t) - 3$ then $g = hx_1x_2x_3x'$. Again by using Cube Explorer one can verify that for all $x_1 \in Y, x_2$ and $x_3 \in X_f$ only the following cases are possible:

   (a) $d(hx_1) \leq 17$;
   (b) $d(hx_1x_2) \leq 18$;
   (c) $d(hx_1x_2x_3) \leq 19$.

   and in each case $d(g) \leq 19 + d(t) - 3 \leq 28$.

Consequently we can conclude that Rubik's cube is solvable in at most 28 moves.

9 Refinement: Rubik's Cube can be Solved in 27 Moves

Here we apply exactly the same techniques as we did in QTM in Section 6. It turns out that there are only 40 cosets of the group T_H (defined in Section 6) we need to consider here. If we reduce these cosets by M we obtain only 7 cosets $E_1, E_2, E_3, E_4, E_5, E_6, E_7$ with representatives $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ of the group T_H for which we computed distance records in Table 5. No element in these cosets requires more than 20 moves. This shows that Rubik's cube is solvable in 27 moves.

$e_1 := U^{-1}F^2D^{-1}R^2UR^2D^2L^2R^{-1}U^{-1}F^{-1}RDLUB^{-1}UB^2F^{-1}U^{-1}RU^{-1}$

$e_2 := U^{-1}F^2DL^2DR^2D^{-1}LB^2R^2F^{-1}D^2U^{-1}RB^2R^2U^2L^2U^{-1}R^{-1}$

$e_3 := DB^2F^2U^{-1}R^2UL^2FL^{-1}R^{-1}BDUBL^{-1}R^{-1}FR^2UR^2$

$e_4 := D^2B^2U^{-1}R^2F^2UR^2F^{-1}L^2DF^2R^{-1}F^2L^{-1}UL^{-1}B^2D^{-1}F^{-1}R^2$

$e_5 := F^{-1}D^2BFU^2L^{-1}U^{-1}R^2D^{-1}L^{-1}U^2BFD^2F^{-1}DF^2U$

$e_6 := U^2RB^2U^2B^2L^{-1}B^2DL^{-1}BLR^{-1}FR^2UL^{-1}RDR^{-1}F$

$e_7 := D^2F^{-1}U^2B^{-1}U^2F^{-1}R^{-1}D^{-1}BF^{-1}R^2F^2D^{-1}RU^{-1}L^2B^2FL^2R^{-1}$

Table 5:

dist	d.r. of E_1	d.r. of E_2	d.r. of E_3	d.r. of E_4	d.r. of E_5	d.r. of E_6	d.r. of E_7
14	0	0	64	16	64	0	0
15	720	552	1204	512	1040	624	496
16	14508	13722	22362	12452	18338	16050	10896
17	226036	241910	354750	265113	312168	279938	197241
18	2120898	2504080	2610544	2611642	2549550	2406252	2218817
19	1618793	1221024	992332	1091498	1100104	1278378	1553826
20	357	24	56	79	48	70	36

As for the quarter turn metric one can easily verify that the group elements appearing in these cosets have distance less than 20 by providing generator expressions for each element of the cosets.

10 Conclusion

In this paper we have introduced upper bounds for Rubik's cube in QTM and FTM. The correctness proofs of our bounds can be obtained by independent verifications with computer algebra systems like GAP. Namely, one just takes as input the described representation of group elements which can be produced by the author on demand.

We expect that further improvements of the bounds are possible using the above techniques together with additional computational power. Compared to the approach in [Kunkle and Cooperman, 2007] we are significantly faster and expect that the group H which Michael Reid first used for upper bounds is most suitable for giving better bounds comparing with the group $\langle U^2, L^2, F^2, B^2, R^2, D^2\rangle$ used by Daniel Kunkle and Gene Cooperman.

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